# MATH 1A - MIDTERM 2 - SOLUTIONS

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1. *(15 points)* Using **the definition** of the derivative, calculate $f'(4)$, where:

$$f(x) = \sqrt{x}$$

$$\begin{aligned} f'(4) &= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \\ &= \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} \\ &= \lim_{x \to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} \\ &= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} \\ &= \frac{1}{2 + 2} \\ &= \frac{1}{4} \end{aligned}$$

Hence, $\boxed{f'(4) = \frac{1}{4}}$

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2. *(15 points)* Using **the definition** of the derivative, calculate $f'(x)$, where:

$$f(x) = x^2 + x$$

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x\to a} \frac{x^2 + x - (a^2 + a)}{x - a} \\
&= \lim_{x\to a} \frac{x^2 + x - a^2 - a}{x - a} \\
&= \lim_{x\to a} \frac{x^2 - a^2 + (x - a)}{x - a} \\
&= \lim_{x\to a} \frac{x^2 - a^2}{x - a} + \frac{x - a}{x - a} \\
&= \lim_{x\to a} \frac{(x - a)(x + a)}{x - a} + \lim_{x\to a} \frac{x - a}{x - a} \\
&= \lim_{x\to a} x + a + \lim_{x\to a} 1 \\
&= 2a + 1
\end{aligned}$$

Hence, $\boxed{f'(x) = 2x + 1}$

**Other solution:**

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{(a+h)^2 + (a+h) - (a^2+a)}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 + a + h - a^2 - a}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2 + h}{h} \\
&= \lim_{h \to 0} \frac{h(2a+h+1)}{h} \\
&= \lim_{h \to 0} 2a + h + 1 \\
&= 2a + 1
\end{aligned}
$$

Hence, $\boxed{f'(x) = 2x + 1}$

3. *(50 points, 5 points each)* Find the derivatives of the following functions:

(a) $f(x) = e^x + \cos(x) + 1$

$$\boxed{f'(x) = e^x - \sin(x)}$$

(b) $f(x) = x\ln(x) - x$

$$f'(x) = \ln(x) + x\left(\frac{1}{x}\right) - 1 = \ln(x) + 1 - 1 = \ln(x)$$

(c) $f(x) = \frac{e^x}{(\sin(x))^2}$

$$f'(x) = \frac{e^x\left(\sin(x)\right)^2 - e^x 2\sin(x)\cos(x)}{\left(\sin(x)\right)^4} = \frac{e^x(\sin(x) - 2\cos(x))}{\left(\sin(x)\right)^3}$$

(d) $f(x) = \sqrt{\ln(x^2+1)}$

$$\boxed{f'(x) = \left(\frac{1}{2\sqrt{\ln(x^2+1)}}\right)\left(\frac{1}{x^2+1}\right)(2x)}$$

**Note:** Woe to you if you tried to simplify!

$$\ln(x^2+1) \neq \ln(x^2) + \ln(1)$$

(e) $f(x) = \tan(\tan(\tan(x)))$

$$f'(x) = \sec^2(\tan(\tan(x))) \times \sec^2(\tan(x)) \times \sec^2(x)$$

(f) $f(x) = (\sin(x))^x$

Let $y = (\sin(x))^x$

1) $\ln(y) = x\ln(\sin(x))$

2) $\frac{y'}{y} = \ln(\sin(x)) + x\frac{\cos(x)}{\sin(x)}$

3) $y' = y\left(\ln(\sin(x)) + x\frac{\cos(x)}{\sin(x)}\right)$

$\boxed{y' = (\sin(x))^x\left(\ln(\sin(x)) + x\frac{\cos(x)}{\sin(x)}\right)}$

(g) $y'$, where $x^2 + 3xy + y^2 = 1$

Differentiating: $2x + 3y + 3xy' + 2yy' = 0$

Solving for $y'$: $y'(3x + 2y) = -(2x + 3y)$

$$y' = \frac{-(2x + 3y)}{3x + 2y}$$

(h) The equation of the tangent line to $y = x^4 + 3x$ at the point (1,4)

$y' = 4x^3 + 3$

Slope $= y'(1) = 4 + 3 = 7$

Equation: $\boxed{y - 4 = 7(x - 1)}$

(i) The equation of the tangent line at $(1, 0)$ to the curve:

$$\sin(y^2 + x\sin(y)) = x^2 + 1$$

Differentiating:

$$\cos(y^2 + x\sin(y))(2yy' + \sin(y) + x\cos(y)y') = 2x$$

If $x = 1$ and $y = 0$, we get: $\cos(0)(0 + 0 + y') = 2$, so $y' = 2$

Equation: $y - 0 = 2(x - 1)$, so $\boxed{y = 2x - 2}$

(j) $\boxed{f''(x)}$, where $f(x) = \tan^{-1}(x)$

$f'(x) = \frac{1}{1+x^2}$

$f''(x) = -\frac{1}{(1+x^2)^2}(2x) = -\frac{2x}{(1+x^2)^2}$

4. *(20 points)* Show that the sum of the $x-$ and $y-$ intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is c.

Slope:

$$\begin{aligned}\frac{1}{2\sqrt{x}}+y'\left(\frac{1}{2\sqrt{y}}\right)=&0\\ y'\left(\frac{1}{2\sqrt{y}}\right)=&-\frac{1}{2\sqrt{x}}\\ y'=&-\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}\\ y'=&-\frac{2\sqrt{y}}{2\sqrt{x}}\\ y'=&-\frac{\sqrt{y}}{\sqrt{x}}\end{aligned}$$

Equation: At $(x_0,y_0)$, the slope is $-\frac{\sqrt{y_0}}{\sqrt{x_0}}$, and so the equation of the tangent line at $(x_0,y_0)$ is:

$$y-y_0=-\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$$

$y-$intercept:
To find the $y-$intercept, set $x=0$ and solve for $y$:

$$\begin{aligned}y-y_0=&-\frac{\sqrt{y_0}}{\sqrt{x_0}}(0-x_0)\\ y-y_0=&-\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)\\ y-y_0=&\sqrt{y_0}\sqrt{x_0}\\ y=&y_0+\sqrt{y_0}\sqrt{x_0}\end{aligned}$$

$x-$intercept:
To find the $x-$intercept, set $y=0$ and solve for $x$:

$$\begin{aligned}0 - y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)\\ -y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)\\ x - x_0 &= -\frac{\sqrt{x_0}}{\sqrt{y_0}}(-y_0)\\ x &= x_0 + \sqrt{x_0}\sqrt{y_0}\end{aligned}$$

Sum:
The sum of the $y-$ and $x-$ intercepts is:

$$(y_0 + \sqrt{y_0}\sqrt{x_0}) + (x_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0$$

Using the hint:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0})^2 + 2\sqrt{x_0}\sqrt{y_0} + (\sqrt{y_0})^2 = (\sqrt{x_0} + \sqrt{y_0})^2$$

But since $(x_0, y_0)$ is on the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$, we get $\sqrt{x_0} + \sqrt{y_0} = \sqrt{c}$.

And so, finally we get that the sum of the $x-$ and $y-$ intercepts is:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c$$

**Bonus 1** *(5 points)* Assume $f$ is a **nonzero** function which satisfies:

$$\begin{cases} f'(x) = & f(x) \\ f(0) = & 1 \end{cases}$$

For this problem, the following property might be useful:

**Property:** If $g'(x) = 0$ for all $x$, then $g(x) = C$, where $C$ is a constant.

(a) Show that $f(x+a) = f(x)f(a)$.

**Hint:** Define $g(x) = \frac{f(x+a)}{f(x)}$

$$g'(x) = \frac{f'(x+a)f(x) - f(x+a)f'(x)}{(f(x))^2} = \frac{f(x+a)f(x) - f(x+a)f(x)}{(f(x))^2} = 0$$

(We used the fact that $f'(x+a) = f(x+a)$ and $f'(x) = f(x)$)

So $g'(x) = 0$, so $g(x) = C$.

To find out what $C$ is, notice that:

$$C = g(0) = \frac{f(a)}{f(0)} = f(a)$$

Hence $C = f(a)$, and we get that $g(x) = f(a)$, so

$$\frac{f(x+a)}{f(x)} = f(a)$$

And multiplying by $f(x)$, we get:

$$f(x+a) = f(x)f(a)$$

(b) Show that $f(-x) = \frac{1}{f(x)}$

**Hint:** Define $g(x) = f(-x)f(x)$.

$$g'(x) = -f'(-x)f(x) + f(-x)f'(x) = -f(-x)f(x) + f(-x)f(x) = 0$$

Hence $g(x) = C$.

But:

$$C = g(0) = f(0)f(0) = 1 \times 1 = 1$$

So $g(x) = 1$, and $f(-x)f(x) = 1$, and:

$$f(-x) = \frac{1}{f(x)}$$

(c) Show that $f(ax) = f(x)^a$

**Hint:** Define $g(x) = \frac{f(ax)}{f(x)^a}$

$$\begin{aligned}
g'(x) =& \frac{f'(ax)a(f(x))^a - f(ax)af(x)^{a-1}f'(x)}{f(x)^{2a}} \\
=& \frac{af(ax)f(x)^a - af(ax)f(x)^{a-1}f(x)}{f(x)^{2a}} \\
=& \frac{af(ax)f(x)^a - af(ax)f(x)^a}{f(x)^{2a}} \\
=& 0
\end{aligned}$$

So $g(x) = C$, but:

$$C = g(0) = \frac{f(0)}{f(0)^a} = \frac{1}{1} = 1$$

So $g(x) = 1$, so $\frac{f(ax)}{f(x)^a} = 1$, so:

$$f(ax) = f(x)^a$$

(d) *(2 extra points)* In fact, the converse statement is true too! Namely, if $f$ is a function with:

$$\begin{cases} f(a+b) = & f(a)f(b) \\ f(0) = & 1 \\ f'(0) = & 1 \end{cases}$$

Show that $f(x) = e^x$.

Notice that we're not making any assumptions about $f$ being smooth, only that it is differentiable at $0$!

**Hint:** All you need to show is that $f'(x) = f(x)$ for all $x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= \lim_{h\to 0} f(x)\frac{f(h) - 1}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} \\ &= f(x)f'(0) \\ &= f(x) \end{aligned}$$

**Bonus 2** *(5 points)* The following bonus problem is meant to show you that derivatives can behave in very strange ways!

(a) Find an example of a function $f$ with $\lim_{x\to\infty} f(x) = 0$ but $\lim_{x\to\infty} f'(x) \neq 0$ (in fact, the limit does not exist)

Let $\boxed{f(x) = \frac{\sin(x^2)}{x}}$.

Then $\lim_{x\to\infty} f(x) = 0$ by the Squeeze Theorem.

But:

$$f'(x) = \frac{\cos(x^2)(2x)x - \sin(x^2)}{x^2} = 2\cos(x^2) - \frac{\sin(x^2)}{x^2}$$

Now $\lim_{x\to\infty} \frac{\sin(x^2)}{x^2} = 0$ by the squeeze theorem, but $\lim_{x\to\infty} \cos(x^2)$ does not exist, from which it follows that:
$\lim_{x\to\infty} f'(x)$ does not exist! (so in fact it is $\neq 0$)

(b) Find an example of a function $f$ that is differentiable at $0$, but whose derivative is not continuous at $0$.

Let $\boxed{f(x) = x^2 \sin(\frac{1}{x})}$.

Then:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

(by the squeeze theorem)

So $f$ is differentiable at $0$ with $f'(0) = 0$.

However:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

Now $\lim_{x\to 0} 2x\sin(\frac{1}{x}) = 0$ (by Squeeze Thm), but $\lim_{x\to 0} -\cos\left(\frac{1}{x}\right)$ does not exist, so $\lim_{x\to 0} f'(x)$ does not exist, and in fact does not equal to $f'(0) = 0$.

So $\lim_{x\to 0} f'(x) \neq f'(0)$, hence $f'$ is not continuous at 0.